Counterexamples to theorems published and proved in recent literature on Clifford algebras, spinors, spin groups and the exterior algebra

and preliminary discussions on R, C, H and O, the cross product, objects in 4D, rotations in 4D and the Maxwell equations in Cl_3.

William K. Clifford (1845-1879)

MOTTO: *In research, counterexamples show us that we are going the wrong way. They tell us where not to go in exploring a new domain.*

This web-page is an abridgement of my paper article on counterexamples, published in a peer-reviewed journal. The paper discusses the role of counterexamples in scientific progress, in evaluating validity of theories. Counterexamples falsify theories. In mathematics, **theorems are provable statements**, which cannot be falsified, by definition. However, it is possible to falsify statements published as theorems in mathematical literature.

I give a list of **counterexamples, which satisfy all the assumptions of the alleged theorems without the conclusions being valid** I show that there are many false statements published as theorems, in the mathematical literature. In other words, I falsify purported theorems.

The job of a mathematician is to prove statements. Proofs are evaluated with a few colleagues and published in a refereed journal. The purpose of publication is to expose the theorems to public scrutiny. There follows a public debate, which might result in a revision of a theorem (or rather in a revision of a manifestation of the theorem in the literature).

Mathematics is universal and effectively applied to the real world. This often leads mathematicians to a **cognitive illusion**: when several members of a research group have accepted a new statement as a theorem, the statement becomes true and unfalsifiable (in the minds of the members of the research group). I show that such a collective position is untenable: I falsify statements, which groups of renowned mathematicians have labeled as theorems. Most of the renowned mathematicians are still living and can participate in the evaluation of possible validity of my counterexamples to their "theorems".

The mistakes occurred at frontiers of joint explorations of mathematicians, who still had inaccurate cognitive charts of the new domains they were exploring. Young scientists with recent PhD's are reluctant to admit their mistakes, because for them it is important to get a position. On the other hand, senior scientists also become reluctant to admit their mistakes toward the end of their careers, because they fear that their life works will collapse if mistakes are detected. In general, two groups are ready to admit their blunders: those far removed from the frontline of research and those who produce new knowledge. Producers of new knowledge know the factors of uncertainty: closer to the source of knowledge, the uncertainty increases. The transmitters and users of knowledge are more certain about their theories: university teachers and engineers are supposed to know the facts.

Some mistakes were confined to individuals, who could be easily convinced about their mistakes. Some mistakes were common to several mathematicians, or groups. Such **groups of mathematicians had a collective cognitive illusion about the "mathematical reality"**, which they described and studied by means of a poor language. To break such a cognitive illusion, I often had to learn the poor language and culture of the mistaken groups. Sometimes such groups defended their cognitive bugs vigorously.

The falsified "theorems" were seldom used in subsequent deductions and did not have an impact on the works of other researchers, because I often presented my counterexamples soon after the first occurrence of the "theorems". Nevertheless, other researchers often repeated the same mistakes independently. From this observation, I come to the following main result of my findings:

> **Creative research mathematicians, exploring the frontiers of our common body of knowledge, tend to make similar mathematical mistakes.**

This leads to collective cognitive bugs, among advanced researchers. In the realm of such bugs, false statements collectively hold true, cannot be distinguished from the correct ones. In other words, there is no way to make a distinction between

1. a theorem, and
2. a statement labeled as a theorem by all experts.

If there were, experts could just agree on classifying all statements labeled as theorems into the above two classes. Mathematicians have responded to this observation by the following comments:

1. a theorem cannot be falsified, since it is a provable statement, and
2. if a statement has a counterexample it cannot be a theorem.

The responses overlook cultural aspects of theorem-making: theorems held true today may well become false in the future, if language becomes more precise and resolution of the research topic enhances.

I informed almost all of the mathematicians about their errors, prior to exhibition of this web-page. The mathematicians have mostly admitted their mistakes, after a reasoned dialogue, lasting for a few months or sometimes years. The course of events was usually as follows: I find that a theorem does not hold and work out the simplest non-trivial counterexample. **I pay special attention to interpreting the text in the way the author has intended so as to make sure that my counterexample reveals an interior inconsistency**. Then I send a letter to the author enclosing a detailed description of my counterexample. To avoid misinterepretation of my intentions, I also enclose a photocopy of the theorem in the envelope, underline false parts in red and mark in the margin the word WRONG, in red.

The authors usually defend their theorems, at first. After a few letters have been exchanged, most of the authors accept validity of my counterexamples and admit their mistakes. When the mistake has been understood, the author usually explains the error away as casual and insignificant.

Some of the counterexamples stem from the failure of the authors to list all the assumptions of their theorems or, more serioulsy, to check their statements for small numbers of indices or in low dimensions (typically 2,3). Quite a few of the counterexamples consist of exceptional cases in lower dimensions (typically 2,4,7,8). Counterexamples are also given in the cases where the author failed to notice a general pattern after some dimension (typically at and above 4,5,6). In some cases, the authors had just poorly designed or chosen concepts, which require exceptional cases, like the "inner product", a symmetrized contraction imitating a derivation.

Informing colleagues about their own errors is more subtle. It offers them an opportunity to learn more mathematics. It might also result in a feeling of insufficiency, a cognitive conflict, and instigate a learning process, and thus indirectly lead into the same final situation, namely cognitive growth. See Ginsburg & Opper 1988.

In order to benefit from mathematical arguments presented on this web-page, the viewer should have references at hand, and follow the reasoning of the counterexamples line by line. At first, we have to fix some preliminary notations, since some viewers might be unfamiliar with Clifford algebras.

The Clifford algebra of the Euclidean plane

Consider the Euclidean plane $\mathbf{R}^2$ with a quadratic form sending a vector $x\mathbf{e}_1+y\mathbf{e}_2$ to the scalar x^2+y^2. The Clifford algebra Cl_2 of $\mathbf{R}^2$ is a real associative algebra of dimension 4 with unit element 1. It contains copies of $\mathbf{R}$ and $\mathbf{R}^2$ in such a way that the square of the vector $x\mathbf{e}_1+y\mathbf{e}_2$ equals the scalar x^2+y^2, as an equation

$$(x\mathbf{e}_1+y\mathbf{e}_2)^2 = x^2+y^2.$$

It follows that Cl_2 has a basis $(1,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_{12})$, and the following multiplication rules are satisfied: the orthogonal unit vectors $\mathbf{e}_1,\mathbf{e}_2$ in $\mathbf{R}^2$ square up to 1

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$$

and anticommute

$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$$

and their product equals $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$. Computing the square of $\mathbf{e}_{12}$ we find

$$\mathbf{e}_{12}^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1^2\mathbf{e}_2^2 = -1.$$

Thus, the basis element $\mathbf{e}_{12}$ cannot be a scalar (in $\mathbf{R}$) nor a vector (in $\mathbf{R}^2$), not even a linear combination of a scalar and a vector. In other words, it is a new kind of object, called a *bivector*. The basis elements

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$\mathbf{e}_1,\mathbf{e}_2$
$\mathbf{e}_{12}$

span, respectively, the subspaces of scalars, vectors and bivectors (in Cl_2).

The Clifford algebra Cl_2 has a faithful matrix image, the matrix algebra Mat(2,$\mathbf{R}$) of 2x2-matrices with entries in $\mathbf{R}$. The basis elements $1,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_{12}$ of Cl_2 can be represented by the matrices

$$E_0 = \begin{matrix}1 & 0\\ 0 & 1\end{matrix},\quad E_1 = \begin{matrix}1 & 0\\ 0 & -1\end{matrix},\quad E_2 = \begin{matrix}0 & 1\\ 1 & 0\end{matrix},\quad E_{12} = \begin{matrix}0 & 1\\ -1 & 0\end{matrix}.$$

In comparing Cl_2 and Mat(2,$\mathbf{R}$), which are isomorphic as associative algebras, it should be noted that Cl_2 has more structure: in the Clifford algebra Cl_2 there is a distinguished subspace, isometric to the Euclidean plane $\mathbf{R}^2$. No such privileged subspace exist in Mat(2,$\mathbf{R}$).

The Clifford algebra of the Minkowski space-time

The Clifford algebra $Cl_{3,1}$ of the Minkowski space-time $\mathbf{R}^{3,1}$, with quadratic form $x^2+y^2+z^2-c^2t^2$, is a real associative algebra of dimension 16 with unit element 1. It has a basis consisting of the elements

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$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$
$\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}$
$\mathbf{e}_{123}, \mathbf{e}_{124}, \mathbf{e}_{134}, \mathbf{e}_{234}$
$\mathbf{e}_{1234}$

which span, respectively, the subspaces of scalars, vectors, bivectors, 3-vectors and 4-vectors. The vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ satisfy the following multiplication rules: they are unit vectors with squares

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1 \text{ and } \mathbf{e}_4^2 = -1$$

and they anticommute

$$\mathbf{e}_i\,\mathbf{e}_j = -\mathbf{e}_j\,\mathbf{e}_i \text{ for } i \text{ not } j.$$

We denote

$$\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j \text{ for } i \text{ not } j\text{, and for instance } \mathbf{e}_{1234} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4.$$

These rules and conventions already fix the computation rules of $Cl_{3,1}$.

Example. $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_1^2\,\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_{23}$.

The Clifford algebra $Cl_{3,1}$ of the Minkowski space-time $\mathbf{R}^{3,1}$ is isomorphic, as an associative algebra, to the real 4x4-matrix algebra Mat(4,**R**). This isomorphism allows us to view $Cl_{3,1}$ through its faithful matrix image Mat(4,**R**).

For the convenience of viewers unfamiliar with Clifford algebras, I shall present the first counterexamples by means of a matrix algebra, namely Mat(4,**R**), and then translate the presentation into the corresponding Clifford algebra $Cl_{3,1}$.

Clifford algebra viewed by means of the matrix algebra

The orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ of $\mathbf{R}^{3,1}$ can be represented by the matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

satisfying the multiplication rules

$$E_1^2 = E_2^2 = E_3^2 = I,\ E_4^2 = -I \text{ and } E_iE_j = -E_jE_i \text{ for } i \text{ not } j.$$

Take an element $a = (1+\mathbf{e}_1)(1+\mathbf{e}_{234})$ in $Cl_{3,1}$, represented by the matrix

$$A = (I+E_1)\,(I+E_2E_3E_4) = \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix}.$$

The so-called Clifford-conjugation sending a in $Cl_{3,1}$ to a^- corresponds in Mat(4,**R**) to the anti-automorphism sending A to

$$A^- = E_4A^TE_4^{-1} = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}.$$

Compute the products of A and A^- in different orders to find:

$$A^-A = 0 \quad \text{although} \quad AA^- = \begin{matrix} 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 \end{matrix} \quad \text{is not zero.}$$

In fact, AA^- is not even diagonal, that is, it is not a scalar multiple of I.

After this excursion into matrix algebras the viewer is hopefully prepared for Clifford algebras. Next, I will present some preliminary counterexamples by rewriting the above observation in terms of the Clifford algebra $Cl_{3,1}$.

Preliminary counterexamples in Clifford algebras

Consider the Clifford algebra $Cl_{3,1}$ = Mat(4,**R**) of the Minkowski space-time $\mathbf{R}^{3,1}$. Take an element

$$a = (1+\mathbf{e}_1)(1+\mathbf{e}_{234}) = 1+\mathbf{e}_1+\mathbf{e}_{234}+\mathbf{e}_{1234},$$

and apply Clifford-conjugation (the anti-automorphism of $Cl_{3,1}$ extending the map $\mathbf{x}$ -> -$\mathbf{x}$ in $\mathbf{R}^{3,1}$)

$$a^- = (1+\mathbf{e}_{234})(1-\mathbf{e}_1) = 1-\mathbf{e}_1+\mathbf{e}_{234}+\mathbf{e}_{1234}.$$

Compute the products of a and a^- in different orders to find:

$a^-a = 0$ although $aa^- = 4(\mathbf{e}_{234}+\mathbf{e}_{1234})$ is not zero, not even a scalar in **R**.

Harvey 1990 claims on p. 202, ll. 1, 4-5, in Lemma 10.45, that the following statements are equivalent: (c) aa^- e **R**, (d) a^-a e **R**. Compare the above result to the Lemma, claimed to have been proven by Harvey, and you have a counterexample to Harvey's lemma. In other words, my counterexample falsifies a result of Harvey 1990, Lemma 10.45, (c,d), p. 202, since $a^-a = 0$ is in **R** but $aa^- = 4(\mathbf{e}_{234}+\mathbf{e}_{1234})$ is nonzero and not in **R**. (Harvey introduces the Clifford-conjugation a^- on p. 183; he calls it a hat involution and denotes by $a^\wedge$.)

Gilbert & Murray 1991 denote D(x) = x^-x and prove in Theorem 5.16 that for x such that D(x) is in **R**, it necessarily follows that D(x^-) = D(x) [p. 41, l. 19] and in particular that D(x) = 0 forces D(x^-) = 0 [p. 42, ll. 2-3]. Choose $x = a$ to find D(a) = 0 in **R**, although D(a^-) = $(a^-)^-a^-$ = $4(\mathbf{e}_{234}+\mathbf{e}_{1234})$ is not 0, and thereby not in **R**. Compare this result to Theorem 5.16, claimed to have been proven by Gilbert & Murray, and you have a counterexample to Gilbert & Murray's theorem. In other words, Gilbert & Murray's Theorem 5.16, stating that D(x^-) = D(x), has been falsified by my counterexample. (Gilbert & Murray's conjugation is the Clifford-conjugation, see p. 17.)

The element $x = \mathbf{e}_1+\mathbf{e}_{23}$ in Cl_3 serves as a counterexample to Knus 1991, p. 228, l. 13, since $x^-x = -2\mathbf{e}_{123}$ is not in Cl_3^+. (Knus introduces the Clifford-conjugation x^- on p. 195 and calls it the standard involution s(x). Knus could defend himself by arguing that x in m(x) = x^-x is homogeneous as in l. 3, p. 228.)

For $x = \mathbf{e}_1+\mathbf{e}_{23}$ in Cl_3, $x^-x = -2\mathbf{e}_{123}$ is not in **R**, and we have a counterexample to Dabrowski 1988, p. 7, l. 12, who observed his error [see the errata sheet distributed along with his monograph]. In the Clifford algebra Cl_3 of the Euclidean space $\mathbf{R}^3$ there are elements whose exponentials are vectors, like $\mathbf{e}_3 = \exp[(p/2)(\mathbf{e}_{12}-\mathbf{e}_{123})]$. Therefore, for the multivalued inverse of the exponential,

$$\log \mathbf{e}_3 = (p/2)(\mathbf{e}_{12}-\mathbf{e}_{123}).$$

This shows that vectors can have logarithms in a Clifford algebra, and serves as a counterexample to Hestenes 1986, p. 75 [the error is corrected in Hestenes 1987].

All the above counterexamples are trivial, in the sense that an expert reader recognizes the mistakes at the first reading, except maybe the last one. The detection of the last mistake, concerning functions in Clifford algebras, requires knowledge of idempotents, nilpotents and minimal polynomials. A good place to start studying them is Sobczyk, 1997.

Counterexamples about spin groups

The Lipschitz group $\mathbf{L}_{p,q}$, also called the Clifford group although invented by Lipschitz 1880/86, could be defined as the subgroup in $Cl_{p,q}$ generated by invertible vectors $\mathbf{x}$ e $\mathbf{R}^{p,q}$, or equivalently by either of the following ways

$\mathbf{L}_{p,q}$ = {s e $Cl_{p,q}$; for all $\mathbf{x}$ e $\mathbf{R}^{p,q}$, $s\mathbf{x}s^{\wedge-1}$ e $\mathbf{R}^{p,q}$},
$\mathbf{L}_{p,q}$ = {s e $Cl_{p,q}^+$ \cup $Cl_{p,q}^-$; for all $\mathbf{x}$ e $\mathbf{R}^{p,q}$, $s\mathbf{x}s^{-1}$ e $\mathbf{R}^{p,q}$}.

Note the presence of the grade involution: s -> $s^\wedge$ (the automorphism of $Cl_{p,q}$ extending the map $\mathbf{x}$ -> -$\mathbf{x}$ in $\mathbf{R}^{p,q}$), and/or restriction to the even/odd parts $Cl_{p,q}^\pm$. The Lipschitz group $\mathbf{L}_{p,q}$ has a subgroup, normalized by the reversion: s -> $s^\sim$ (the anti-automorphism of $Cl_{p,q}$

extending the identity map **x** -> **x** in $\mathbf{R}^{p,q}$),

$$\mathbf{Pin}(p,q) = \{s \in \mathbf{L}_{p,q};\ ss^\sim = \pm 1\},$$

with an even subgroup

$$\mathbf{Spin}(p,q) = \mathbf{Pin}(p,q) \cap Cl_{p,q}^+,$$

which contains as a subgroup the two-fold cover

$$\mathbf{Spin}_+(p,q) = \{s \in \mathbf{Spin}(p,q);\ ss^\sim = 1\}$$

of the connected component $SO_+(p,q)$ of $SO(p,q) \subset O(p,q)$.

Although $SO_+(p,q)$ is connected, its two-fold cover $\mathbf{Spin}_+(p,q)$ need not be connected. In particular,

$$\mathbf{Spin}_+(1,1) = \{x+y\mathbf{e}_{12};\ x,y \in \mathbf{R},\ x^2-y^2 = 1\}$$

has two components, two branches of a hyperbola [and so the group

$$\mathbf{Spin}(1,1) = \{x+y\mathbf{e}_{12};\ x,y \in \mathbf{R},\ x^2-y^2 = \pm 1\}$$

has four components]. This serves as a counterexample to Choquet-Bruhat et al. 1989, p. 37, ll. 2-3, p. 38, ll. 22-23 [see also p. 27, ll. 4-5]. Although the two-fold covers $\mathbf{Spin}(n) = \mathbf{Spin}(n,0) \simeq \mathbf{Spin}(0,n)$, $n > 2$, and $\mathbf{Spin}_+(n-1,1) \simeq \mathbf{Spin}_+(1,n-1)$, $n > 3$, are simply connected, $\mathbf{Spin}_+(3,3)$ is not simply connected, and therefore not a universal cover of $SO_+(3,3)$, since the maximal compact subgroup $SO(3)\times SO(3)$ of $SO_+(3,3)$ has a four-fold universal cover $\mathbf{Spin}(3)\times\mathbf{Spin}(3)$. The two-fold cover $\mathbf{Spin}_+(3,3)$ of $SO_+(3,3)$ is doubly connected, contrary to the claims of Lawson & Michelsohn 1989, p. 57, l. 22, and Göckeler & Schücker 1987, p. 190, l. 17.

Lawson & Michelsohn 1989 give also correct information about the connectivity properties of the rotation groups $SO_+(p,q)$, see p. 20, ll. 6-8.

As a consequence, $\mathbf{Spin}_+(3,3) \simeq SL(4,\mathbf{R})$, and so $\mathbf{Spin}_+(3,3)/\{\pm 1\} \not\simeq SL(4,\mathbf{R})$ contrary to the claims of Harvey 1990, p. 272, l. 24, and Lawson & Michelsohn 1989, p. 56, l. 21. Moreover, the element $\mathbf{e}_1\mathbf{e}_2\ldots\mathbf{e}_6 \in \mathbf{Spin}(3,3)\backslash\mathbf{Spin}_+(3,3)$ is not in $\mathbf{Spin}_+(3,3)$, since it is a preimage of $-I \in SO(3,3)\backslash SO_+(3,3)$, contrary to the claims of Lawson & Michelsohn 1989, p. 57, ll. 29-30.

Comment on Bourbaki 1959. The groups $\mathbf{Pin}(p,q)$ and $\mathbf{Spin}(p,q)$, obtained by normalizing the Lipschitz group $\mathbf{L}_{p,q}$, are two-fold coverings of the orthogonal and special orthogonal groups, $O(p,q)$ and $SO(p,q)$, respectively. If one defines, instead of the Lipschitz group, a slightly different group

$$\mathbf{G}_{p,q} = \{s \in Cl_{p,q};\ \text{for all } \mathbf{x} \in \mathbf{R}^{p,q},\ s\mathbf{x}s^{-1} \in \mathbf{R}^{p,q}\},$$

one obtains, only in even dimensions, a cover of $O(p,q)$. Furthermore, for odd $n=p+q$, an element of $\mathbf{G}_{p,q}$ need not be even or odd, but might have an inhomogeneous central factor $x+y\mathbf{e}_{12\ldots n} \in \mathbf{R}+\wedge^n\mathbf{R}^{p,q}$. Thus Bourbaki 1959, p. 151, Lemme 5, does not hold, as has been observed by Deheuvels 1981, p. 355, Moresi 1988, p. 621, and by Bourbaki himself [see Feuille d'Errata No. 10 distributed with Chapters 3,4 of Algèbre Commutative 1961].

The confusion about proper covering of $O(p,q)$ in $Cl_{p,q}$ pops up frequently.

In the Lipschitz group every element $s \in \mathbf{L}_{p,q}$ is of the form $s = r\,g$, where $r \in \mathbf{R}\backslash\{0\}$, $g \in \mathbf{Pin}(p,q)$. The group $\mathbf{G}_{p,q}$ does not have this property in odd dimensions. For instance, the central element $z=x+y\mathbf{e}_{123} \in Cl_3$, with non-zero $x,y \in \mathbf{R}$, satisfies $z \in \mathbf{G}_3$, but $z \neq r\,g$, $g \in \mathbf{Pin}(3)$. This serves as a counterexample to Baum 1981, p. 57, l. -1. [Baum's $C_{n,k}$ means $Cl_{k,n-k}$, see p. 51, and her $Pin(n,k)$ means $\mathbf{Pin}(k,n-k)$, see p. 53. Note that the two-fold cover of $O(3)$,

$$\mathbf{Pin}(3) = \mathbf{Spin}(3) \cup \mathbf{e}_{123}\mathbf{Spin}(3) \simeq SU(2) \cup i\,SU(2),$$

is a subgroup of $\mathbf{G}_3$, but since the actions are defined differently, $\mathbf{G}_3$ does not cover $O(3)$.]

For all $s \in \mathbf{G}_3$, $ss^\sim > 0$. Therefore, if we normalize $\mathbf{G}_3$ by the reversion, the central factor is not eliminated, but instead we get the group $\{s \in \mathbf{G}_3;\ ss^\sim = 1\} \simeq U(2)$, which does not cover $O(3)$ but covers $SO(3)$ with kernel $\{x+y\mathbf{e}_{123};\ x,y \in \mathbf{R},\ x^2+y^2 = 1\} \simeq U(1) \not\simeq \{\pm 1\}$. Compare this to Figueiredo 1994, p. 230, ll. -4.

Exponentials of bivectors. There are two possibilities to exponentiate a bivector $\mathbf{B} \in \wedge^2\mathbf{R}^{p,q}$: the ordinary/Clifford exponential $\exp(\mathbf{B})$, and the exterior exponential $\exp^\wedge(\mathbf{B})$, where the product is the exterior product. If the exterior exponential $\exp^\wedge(\mathbf{B})$ is invertible with respect to the Clifford product, then it is in the Lipschitz group $\mathbf{L}_{p,q}$. For the ordinary exponential we always have $\exp(\mathbf{B}) \in \mathbf{Spin}_+(p,q)$.

All the elements of the compact spin groups $\mathbf{Spin}(n,0) \simeq \mathbf{Spin}(0,n)$ are exponentials of bivectors [when $n > 1$]. Among the other spin groups the same holds only for $\mathbf{Spin}_+(n-1,1) \sim \mathbf{Spin}_+(1,n-1)$, $n > 4$, see M. Riesz 1958/1993 pp. 160, 172. In particular, the two-fold cover $\mathbf{Spin}_+(1,3) \sim SL(2,\mathbf{C})$ of the Lorentz group $SO_+(1,3)$ contains elements which are not exponentials of bivectors: take $(g_0+g_1)g_2$ e $\wedge^2\mathbf{R}^{1,3}$, $[(g_0+g_1)g_2]^2 = 0$, then $-\exp[(g_0+g_1)g_2] = -1-(g_0+g_1)g_2$ \not= $\exp(\mathbf{B})$ for any $\mathbf{B}$ e $\wedge^2\mathbf{R}^{1,3}$.

Note, that in $\mathbf{Spin}_+(4,1) \sim Sp(2,2)$ we have $-\exp((\mathbf{e}_1+\mathbf{e}_5)\mathbf{e}_2) = -1-(\mathbf{e}_1+\mathbf{e}_5)\mathbf{e}_2 = \exp((\mathbf{e}_1+\mathbf{e}_5)\mathbf{e}_2+p\,\mathbf{e}_{34})$.

However, all the elements of $\mathbf{Spin}_+(1,3)$ are of the form $\pm\exp(\mathbf{B})$, $\mathbf{B}$ e $\wedge^2\mathbf{R}^{1,3}$. Therefore, the exponentials of bivectors do not form a group, contrary to a statement of Dixon 1994, p. 13, ll. 8-10.

Every element L of the Lorentz group $SO_+(1,3)$ is an exponential of an antisymmetric matrix, $L = \exp(A)$, $gA^Tg^{-1} = -A$; a similar property is not shared by $SO_+(2,2)$, see M. Riesz 1958/1993, pp. 150-152, 170-171. There are elements in $\mathbf{Spin}_+(2,2)$ which cannot be written in the form $\pm\exp(\mathbf{B})$, $\mathbf{B}$ e $\wedge^2\mathbf{R}^{2,2}$; for instance $\pm\mathbf{e}_{1234}\exp(b\mathbf{B})$, $\mathbf{B} = \mathbf{e}_{12}+2\mathbf{e}_{14}+\mathbf{e}_{34}$, $b > 0$. This serves as a counterexample to Doran 1994, p. 41, l. 26, formula (3.16), and to Doran & Hestenes & Sommen 1993, p. 3650, ll. 16-18, formula (4.9).

Riesz also showed, by the same construction on pp. 170-171, that there are bivectors which cannot be written as sums of simple and completely orthogonal bivectors; for instance $\mathbf{B} = \mathbf{e}_{12}+2\mathbf{e}_{14}+\mathbf{e}_{34}$ e $\wedge^2\mathbf{R}^{2,2}$.

The above mistakes are not serious, in the sense that they could be rectified by stipulating the assertions, although such a correction is not obvious in the last examples. The above counterexamples should be easy to understand also for a non-expert, except maybe the last one by M. Riesz, which does require some knowledge of minimal polynomials of linear transformations. Good places to start studying minimal polynomials are Sobczyk, 1997, and M. Riesz 1958/1993, pp. 150-152, 170-171.

More counterexamples

Not to make this web-page too long, I direct the interested viewer to read my two paper articles. There I give more counterexamples, some of them significant, and non-trivial even for experts in Clifford algebras. Counterexamples are given to theorems of leading experts in Clifford algebras: authors of books, editors of journals, organizers of conferences, key-note speakers, etc. I falsify their statements about spinors, charge conjugation, conformal transformations, exterior algebra, Cayley-Dickson process, etc. Some of these mistakes were serious in the sense that the **authors needed considerable time to rectify their cognitive bugs**. This happened in particular, when a group of mathematicians lived in a **collective cognitive illusion** representing poorly the mathematics to be explored. Such misguided cultures defended vigorously their mistaken positions. However, I will not single out the serious mistakes, because outsiders might draw an unjustified conclusion that the mistaken mathematicians are poor.

How did I locate the errors and construct my counterexamples?

First, in trying to get a picture of what is new in a published work, I find something fishy. Then, **I try make sure that I interpret the text the way the author has intended**. Then, I make sure that there is**interior inconsistency** and that the **author has contradicted himself**. Often, I then checked formulas with CLICAL, a computer program designed for Clifford algebra calculations. Evaluating the left hand side with arbitrary arguments satisfying all the assumptions, and comparing the result to the right hand side, reveals sometimes a discrepancy. The next step is to find the **simplest non-trivial counterexample**, in the lowest dimension and degree and with the smallest number of components. In discussions with authors about the fine points of their works, CLICAL has helped me to follow, verify or disqualify, the arguments presented, and to penetrate into the topic, during the dialogue.

Progress in science via counterexamples

Ideally, scientists publish papers for the purpose of testing and evaluating their ideas in a public scrutiny. This ideal has been obscured by the peer review/refereeing system, which pretends to guarantee correctness of ideas -- prior to a public scrutiny, and the tendency to publish in order to get a position in academia. Traditionally, science has progressed through public debates about new ideas: statements, counterexamples, refined statements and new counterexamples, etc.

In mathematics, proving theorems, finding gaps and errors in the proofs, correcting the theorems, detecting errors in the corrected theorems, etc. is a normal activity. This is even more so in advanced mathematics because our cognitive charts are less accurate in new frontiers of knowledge. See Lakatos 1976.

In evaluating the validity of a mathematical theorem, one should either check every detail of its proof or point out a flaw in the chain of deductions or line of thoughts. After a counterexample has been presented, it is often easier to settle whether it fulfils all the assumptions than to check all the details of the proof. As in science, also in mathematics we are faced with the fact that a single counterexample can falsify a theorem or a whole theory. See Popper 1972.

The role of counterexamples in mathematics has been discussed by Lakatos 1976, Dubnov 1963 and Hauchecorne 1988. Lakatos focuses on the historical development of mathematics and Dubnov on various levels of abstraction. Both restrict themselves to a specific topic within mathematics (like this web-page). Hauchecorne gives counterexamples in almost all branches of mathematics. He also elaborates on virtues of counterexamples in teaching and in research: A theorem often necessitates several hypotheses -- to chart out its domain of applications it is important to become convinced about the relevancy of each hypothesis. This can be done by dropping one assumption at a time, and giving a counterexample to each new "theorem". Counterexamples cannot be ignored on the basis that "they do not treat the general case". Counterexamples are not "exceptions that confirm the rule". **In mathematical research, the**

negation of a theorem, affirmation that it is false, is demonstrated by existence of a case, where all the hypotheses are verified without the conclusion being valid. The mathematical justification for the falsity of a theorem is completed by presenting a counterexample. After verification of the validity of a counterexample, further study in the same line, to rescue the "theorem", at whatever generality, is useless and erroneous activity. Falsification of theorems by verifying counterexamples opens new doors for cognitive growth and acts as an impetus of scientific progress .

There are several books listing counterexamples in various branches of mathematics: Capobianco & Molluzo 1978 (graph theory), Gelbaum & Olmsted 1964 (analysis), Forn\ae ss & Stens\o nes 1987 (several complex variables), Khaleelulla 1982 (vector spaces), Romano & Siegel 1986 (statistics), Steen & Seebach 1970 (topology), Stoyanov 1987 (probability) and Wise & Hall 1993 (real analysis). Similarly as Lakatos, Dubnov and Hauchecorne these authors do not point out errors of contemporary mathematicians. The present web-page differs from those studies in that respect: counterexamples are given to the works of living mathematicians, who can participate in a public debate about possible correctness of counterexamples presented in this web-page.

Some scientists refrain from participating in discussions about errors in published works, presumably because they anticipate a misinterpretation on the part of the author. Some scientists refrain from a public debate on errors, because of their mistaken belief that the peer refereeing system guarantees correctness of published works. Often scientists cannot come up with a suggestion on how to evaluate details of their works [other than the peer review system in abstract journals and academic appointments] -- in this web-page such a method has been suggested/revived: public scrutiny focusing on interior consistency of details [of publications available in scientific libraries -- thus guaranteeing everybody's access to a public debate].

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Released March 1997 (last revised May 2002) 5. Lie Groups and Lie Algebras in Clifford Algebras. 6. Dirac Equation and Spinors in n Dimensions. Acknowledgements. References. arXiv:1709.06608v2 [math-ph] 20 Jan 2018. Clifford algebras and their applications to lie groups and spinors. Dmitry shirokov.Â MSC : 15A66, 22E60, 35Q41 Keywords: Clifford algebra, matrix representations, Lie groups, Lie alge-bras, spin groups, Dirac equation, spinors, Pauli theorem, quaternion type, method of averaging. CONTENTS. Introduction . . . 2 1. Denition of Clifford Algebra . . . 3. 1.1. Clifford Algebra as a Quotient Algebra . . . 3 1.2. mathematical physics, Clifford algebras, spin groups, spinors, Dirac equation, Yang-Mills equations. UDC: 514.744, 517.958.Â D. S. Shirokov, â€œClifford algebras and their applications to Lie groups and spinorsâ€, Lectures, Proceedings of the Nineteenth International Conference on Geometry, Integrability and Quantization (Varna, Bulgaria, June 2 - 7, 2017), eds. IvaÃ¯lo Mladenov and Akira Yoshioka, Avangard Prima, Sofia, Bulgaria, 2018, 11â€“53 , arXiv: 1709.06608. One reason was his life long commitment to epistemology and to politics, which made him strongly opposed to the view otherwise currently hel In 1982, Claude Chevalley expressed three specific wishes with respect to the publication of his Works. First, he stated very clearly that such a publication should include his non technical papers. His reasons for that were two-fold. One reason was his life long commitment to epistemology and to politics, which made him strongly opposed to the view otherwise currently held that mathematics involves only half of a man. As he wrote to G. C. Rota on Novembe Not only are examples more concrete than theorems-and thus more accessible-but they cut across individual theories and make it both appropriate and neces-sary for the student to explore the entire literature in journals as well as texts. Indeed, much of the content of this book was first outlined by under-graduate research teams working with the authors at Saint Olaf College during the summers of 1967 and 1968.Â The search for counterexamples is as lively and creative an activity as can be found in mathematics research.Â The discussion of each example is geared to its general level : what is proved in detail in an elementary example may be assumed without comment in a. more advanced example. In many ways the most useful part of this book for reference may be the.