A Semi-Pop Non Mathematical Tutorial on Hilbert Space in Quantum Mechanics

by Jack Sarfatti

In quantum theory the analog of phase space is Hilbert space. As in Hamilton's concept, a single direction or "ray" in Hilbert space (albeit in different mathematical dress) represents the entire quantum state. But the differences between Hilbert space and phase space are important. Hilbert space is a multi-dimensional complex projective ray space. It was developed by David Hilbert in the early part of the twentieth century for purely mathematical reasons. In the 1930s, John von Neumann consolidated ideas from Bohr, Heisenberg, and Schrödinger and placed the new quantum theory in Hilbert space. Hilbert space, like phase space, can be multi-dimensional and therefore can handle all the possibilities of a quantum system in one convenient package.

Configuration space is the position half of phase space in which the momentum coordinates are left out. The quantum wavefunction is a function. A function can be thought of as a black box in which there are a set of inputs and only one output. Like any function in mathematics it has a range and a domain. The domain of a function is the space of the independent variable(s) or input(s). It turns out that there are many possible domains for the quantum wavefunction which is like that archetypal, very mutable, mythical creature, found in many cultures, able to morph into many shapes. The most useful domain of the complex wavefunction is configuration space in addition to time. For example, if the quantum system consists of N point particles, where N is any positive integer greater than zero, the corresponding configuration space has 3N real dimensions, and the associated phase space has 6N real dimensions. The range of any function is the space of the dependent variable or output. Remember, for a "single-valued function", there is only one output, although there can be many inputs. To review, classical mechanics defines a state as a point in phase space which has both momentum and position real dimensions. Quantum mechanics replaces classical phase space with one-half of it called configuration space and a new structure called Hilbert space.

In Hilbert space, a projective ray, or more simply, a direction, represents the Schrödinger wave function in a frame-independent way. The observable operator that is measured defines the frame of reference. A projective or quantum ray is a mathematical entity that exhibits only direction. Unlike a vector, neither the absolute magnitude (i.e., length of the vector), nor the absolute direction of the quantum ray in Hilbert space is physically important. You can make arbitrary global topological stretches and rotations of all the quantum rays as long as they are the same for all the rays. The only thing that physically matters is the relative directional difference between two total quantum rays representing distinguishable states of the same system. This was first discovered by Dirac. The total ray is represented pictorially by a straight line through the origin of the coordinates of a given frame of reference. The projection of this quantum ray on any single axis of this particular frame of reference is the coordinate relative to that axis. There can be an infinity of these axes. However, each coordinate is not a point on a real line, but is a point in a complex plane. This coordinate point of the quantum ray is a 2 real dimensional-vector with a magnitude and a relative phase, both of which, in contrast to the total ray, are physically significant. Remember, each dimension of the Hilbert space has such a 2D-vector representing the projection of the total ray on to the basic ray for that dimension. This vector is called the quantum probability amplitude for the "choice" ,or eigenvalue, corresponding to the given dimension in quantum Hilbert space.

To review, each axis, or dimension, or basic ray, of Hilbert space has a complex plane associated with it. Furthermore, each axis is itself a basis function with a mutable domain but a fixed range. However, the most useful domain is the configuration space from classical Newtonian mechanics. Hilbert space is not a space of simple points, rather it is a space of functions at a higher level of mathematical abstraction. All the axes are at "right angles" to each other. The idea of "right angles" is generalized into the idea of "orthogonal functions". For example, in Fourier analysis a periodic sound wave is analyzed as a Fourier series into a sum of harmonics oscillating at integer multiples of the fundamental frequency at which the sound repeats itself. Each harmonic is a simple oscillating function in time and is an independent axis in a Hilbert space. All of these different harmonics are at "right angles" to each other in a technical sense that we need not go into in detail here. Not all sound waves are periodic. Similarly, we can analyze any arbitrary aperiodic waveform as a Fourier integral over a real continuum of frequencies rather than a discrete equally spaced lattice of harmonic frequencies as we do for Fourier series. The Hilbert space idea, of a space spanned by linearly independent base functions, is a generalization of Fourier's great idea from the nineteenth century. Unlike points, rays in different directions can be added to give an entirely new direction which is essential for quantum theory. This is the quantum superposition principle. Two total quantum rays pointing in different directions in Hilbert space represent two different quantum states. Only the relative directional difference is physically important. This "directional difference" is itself a complex number. That is, the idea of "direction" in quantum Hilbert space is not the same as the idea of "direction" in physical 3D space.

It is evident that Hilbert space does not represent a void as in Newtonian mechanics; it is not merely adding dimensions to three-dimensional space. Rather, Hilbert space is a mathematical device for arranging pieces of information, with each complex coordinate representing a possibility, or probability amplitude, for a given quantum state that might correspond to a definite eigenvalue for energy, or position, or momentum, or spin, etc. Note, that not all of these observable properties can be definite simultaneously. This is the Heisenberg uncertainty principle. For example, if the position along a direction in space is definite, then the component of the momentum along that same direction in space is indefinite. Position and momentum are incompatible. Similarly, energy and time are incompatible. The angular momentum of a particle relative to a direction in space is incompatible with the angle of rotation of that particle in a plane perpendicular to that direction in space. The number of particles in a state is incompatible with the coherent phase of that state. A coherent wave does not have a definite number of quanta in it.

Let's take a definite example. Suppose we have a single particle and we want to measure its momentum in a given direction in space. There are an infinite number of possible momenta. Each definite observable eigenvalue of momentum, is an independent axis, or dimension, or "choice", with a momentum eigenfunction , or basis function, that is a plane wave travelling along the given direction in physical 3D-space. The infinite set of all such plane waves, of all possible wavelengths, in all possible directions in physical space, forms a momentum frame of reference for the quantum Hilbert space, which in this case is infinitely dimensional. If physical space really is a continuum as in Newton's and Einstein's classical physics, then there is a nondenumerable or noncountable infinity of dimensions to the quantum Hilbert space for the motion of this single particle. If we have quantum gravity, then space is a lattice on the Planck scale of 10 -33 cm and we have a denumerable or countable infinity of dimensions for an open forever-expanding universe. Now, if the ray in this space is not parallel to any such momentum axis, i.e., "basic ray" in general, it does not have a definite value for momentum, so the idea that a ray can be decomposed into a series of basic rays perpendicular, i.e., orthogonal, to each other is crucial to quantum theory. Each of these mutually perpendicular basic rays represents a particular potential behavior of a quantum system. The set of all basic rays for a given property form a frame of reference in Hilbert space. But since all the basic rays are perpendicular to each other, none of them has a component along any others, which is to say that all potential activities are classically distinct or mutually exclusive. Also, the number of dimensions needed in this abstract space corresponds to the number of choices available for the quantum system, and this, as we have just seen, can go to infinity.

Let's take another example, if we are talking about the quantum state of an electron, then each axis, or basic ray, is a possibility of finding an electron at a given position at a certain time t. That is, each possible position in physical 3D space forms an axis or basic ray of the a special frame of reference in the Hilbert space of that single electron. Note that each electron has its own private quantum Hilbert space. When we have more than one electron we must multiply their Hilbert spaces together and we then get the "entangled states" of the Einstein-Podolsky-Rosen effect discussed elsewhere in this book. The electrons lose their individual identity in entangled states. As the total ray representing the quantum state moves and changes direction, the magnitudes and the relative phases of all the coordinates, i.e., possibilities , formed by projecting the total ray on to the basic rays, also change. In the classical view, it was assumed that an electron at any given time would be a vector along one of the axes or basic rays in Hilbert space. That is to say, an electron cannot be in more than one place at a time. But in Hilbert space, that is not true: the electron is in all possible places all the time. Therefore, Hilbert space cannot be describing our familiar everyday world. Rather it is describing the world of the wave function which underlies and permeates our three-dimensional world.

Von Neumann, in his quantum theory of measurement, showed that we must project the total quantum ray to one of the possibilities, or basic rays, in order to create a real event. Consider a unit hypersphere of N real dimensions. The radii of all of these unit hyperspheres is 1. By "dimension" here I mean the number of possible axes of rotation needed to locate a point on the hypercircumference or surface of the hypersphere. N = 0 is a degenerate point at 1 on the real line. N = 1 is a unit circle in a plane. N = 2 is a unit spherical shell in physical 3D space. N = 3 is a unit sphere in 4-dimensional Euclidean space. The number of axes is N + 1 So any point on the N-dimensional hypercircumference has N + 1 real coordinates called direction cosines of which only N are independent because the sum of the squares of all N + 1 direction cosines add up to 1. So we have one equation and N + 1 variables giving N independent real variables. This one equation is simply the Pythagorean theorem that the square of the hypotenuse is the sum of the squares of the legs of a right triangle generalized to N + 1 dimensional right triangles. This math was known to the Ancient Greeks of Classical Athens. Now a convenient restriction of quantum Hilbert space, where we normalize the wavefunctions is a simple generalization of this idea. We simply multiply each real direction cosine by an exponential relative phase factor and we have the complex quantum probability amplitude to measure a definite eigenvalue. Attaching relative phases for each coordinate direction cosine of the unit hypersphere takes us to normalized quantum Hilbert space. The relative phase is observable when we change the experiment and measure a property that is incompatible with the original experiment. For example, if we first do an experiment to measure position, and then do an experiment to measure momentum, with the particle in the same quantum state for each experiment, then the relative phases in the position frame of reference of the first experiment, strongly influence the probabilities for measuring different momenta. That is, the relative phases of the complex direction cosines in a given frame of reference are just as important as are the magnitudes. Although the total quantum state is a ray in which only the direction in Hilbert space is physically observable, the projective coordinates of this ray, onto a frame of reference of basic rays, are two-real dimensional vectors in the complex plane in which both the magnitude and relative direction are important.

The Pythagorean sum of the absolute squares of the, now complex, direction cosines is the total probability to measure something. If the experiment is done correctly this has to be 1 in the ideal limit of perfect detectors. The absolute square of each complex direction cosine is the relative probability to measure a definite eigenvalue of the observable property represented by the given frame of reference in Hilbert space. This is an independent axiom of quantum theory that works very well in the laboratory. In our example, the observable was the position of the electron in physical space. If we had two electrons the observable could be the position in the corresponding configuration space of 6 real dimensions. The absolute square of the complex direction cosine is its product with its own mirror image complex conjugate relative to the real axis in the single complex plane associated with a definite axis or basic ray in Hilbert space.

Hilbert space contains infinite dimensions, but these are not geometric. Rather, each dimension represents a state of possible existence for a quantum system. All possible states coexist and add up to the wave function before a measurement is made or a given possibility is selected. The electron unmeasured is a complicated pattern in an infinite-dimensional Hilbert space; it is not in our three-dimensional space, and its attributes are not clearly defined. That is, in the world of the wave function, the electron does not have inherent properties but has incompletely defined potentialities.

To summarize: Originally, classical mechanics was described by Newton's three laws, and these were geometrically expressed in the three-dimensional Cartesian coordinate system.

Hamilton formulated a new description of classical mechanics which was eventually housed in an infinite-dimensional phase space. In this space, a point represents the entire physical system.

First Dirac, and then Von Neumann set quantum theory in an infinite-dimensional complex ray space called Hilbert space. In this space, a ray, or single direction, represents a quantum state, as does the wave function in Schrödinger's formulation.

Each dimension in Hilbert space represents a possible state for a quantum system, so an unmeasured electron exists as a very complicated pattern.

A particular state is selected by projecting the total quantum ray on to one of the basic rays. The set of basic rays forms a frame of reference to observe some property. Each basic ray represents a possible choice or result of a measurement. The resulting coordinates are quantum probability amplitudes that are points in a complex plane, one plane for each basic ray. Points in the complex plane are two real dimensional vectors with a magnitude and also a direction known as the relative phase. The relative phase is observable when we change the experiment and measure a property that is incompatible with the original experiment.

The wave function description in Hilbert space cannot by itself tell us which state will be selected for reality. The projection, of the total ray onto a basic ray, multiplied by its complex conjugate projection is the real number probability for the transformation of a quantum potentiality into a classical actuality. The different quantum potentialities, which coherently co-exist in quantum reality, are mutually exclusive in the classical reality of our consciousness. A particle cannot be in two places at the same time in classical reality, but it is so in quantum reality. This point is basic to the so-called measurement problem and to our understanding of what underlies our three-dimensional world.

Return to Stapp-Sarfatti

Part 1. Mathematical Foundations of Quantum Mechanics. Chapter 1. Hilbert spaces Â§1.1. Hilbert spaces Â§1.2.Â Some quantum mechanics Self-adjoint operators Quadratic forms and the Friedrichs extension Resolvents and spectra Orthogonal sums of operators Self-adjoint extensions Appendix: Absolutely continuous functions. 55 58 67 73 79 81 84. Chapter 3. The spectral theorem Â§3.1.Â limit is unique if it exists (this is not true for a semi-metric). Every convergent sequence is a Cauchy sequence; that is, for every Îµ > 0 there is some N âˆˆ N such that. d(xn, xm) â‰¤ Îµ, n, m â‰¥ N. $\begingroup$ Actually, the Hilbert space is unique in a mathematical sense: any 2 infinite-dimensional separable Hilbert spaces are isomorphic. $\endgroup$ â€“ DanielC Mar 30 '14 at 15:41. 1. $\begingroup$ But at the same time, generally speaking, different systems will have different Hilbert spaces. A Hilbert space is just a kind of vector space, and the number of dimensions your Hilbert space has is going to correspond (in some sense) to the amount of freedom your system has.Â I think in quantum mechanics we assign to each system a specific Hilbert space i.e. if systems are different then their Hilbert spaces are different. Is this true? If not why? distribution theoryâ€"in Quantum Mechanics, to rig a Hilbert space means simply to equip that Hilbert space with distribution theory. Thus, the RHS is not a replacement but an enlargement of the Hilbert space.Â Such non-invariance makes expectation values, uncertainties and commutation relations not well dened on the whole of H. The space Î¦ is the largest subspace of the Hilbert space on which such expectation values, uncertainties and commutation relations are well dened.Â 2. Motivating the rigged Hilbert space. The linear superposition principle and the probabilistic interpretation of Quantum Mechanics are two major guiding principles in our understanding of the microscopic world. In quantum mechanics, an observable A commuting with the Hamiltonian [H,A] = 0, corresponds to a symmetry of the time-independent Schrodinger equation HÏˆ = EÏˆ. How to we compute the conserved quantity related to A? In particular, what is the the conserved quantity associated with the identity operator?Â In hindsight, Noether's theorem is a dramatic hint of quantum mechanics. Mariano is completely correct in his comment that the conserved quantity is A itself, but it deserves a bit of explanation. A classical probabilistic system is characterized by an algebra of random variables. You could consider the Boolean random variables, in which case the algebra is a σ-algebra Ω. I. 3. Quantum Mechanics of a (Non-Relativistic Spinless) Particle: A. The Hilbert Space Model for a Physical System: states, observables, expected values, purity, & superselection rules B. Observables, (Generalized) Spectral Theorem, and Quantum Probability C. Dynamics, Translations, Rotations, Boosts, and Their Generators D. Symmetry E. Non-relativistic Conserved Current F. Uncertainty Relations G. Informational (In)completeness H. Entropy and Information I. Multiparticle Quantum. Systems J. Quantum Premeasurements & Constraints of Conservation Laws K. The Objectification Problem and