Proving some geometric inequalities by using complex numbers

Titu Andreescu and Dorin Andrica

Abstract

Let $ABC$ be a triangle and let $R$ and $r$ be its circumradius and inradius, respectively. One of the most important result in Triangle Geometry is Euler’s inequality $R \geq 2r$. There are many proofs for this inequality (geometric, trigonometric, analytic etc.). We refer to the books [3] and [4] for some useful discussions on this inequality.

In this note we will give other proofs by using complex numbers. The method of complex numbers in Geometry is a powerful technique. For other applications we refer to our new book [2].

2000 Mathematical Subject Classification: 26D99, 97D50

Theorem 1. Let $P$ be an arbitrary point in the plane of triangle $ABC$. Then

$$\alpha PB \cdot PC + \beta PC \cdot PA + \gamma PA \cdot PB \geq \alpha \beta \gamma,$$

where $\alpha, \beta, \gamma$ are the side lengths of triangle $ABC$. 

19
Proof. Let us consider the origin of the complex plane at \( P \) and let \( a, b, c \) be the affixes of vertices of triangle \( ABC \). From the algebraic identity

\[
\frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-c)(b-a)} + \frac{ab}{(c-a)(c-b)} = 1
\]

by passing to moduli, it follows that

\[
|b||c|\frac{1}{|a-b||a-c|} + |c||a|\frac{1}{|b-c||b-a|} + |a||b|\frac{1}{|c-a||c-b|} \geq 1.
\]

Taking into account that \( |a| = PA, |b| = PB, |c| = PC \) and \( |b-c| = \alpha, |c-a| = \beta, |a-b| = \gamma \), (2) is equivalent to

\[
\frac{PB \cdot PC}{\beta \gamma} + \frac{PC \cdot PA}{\gamma \alpha} + \frac{PA \cdot PB}{\alpha \beta} \geq 1,
\]

i.e. the desired inequality.

Remarks. 1) If \( P \) is the circumcenter \( O \) of triangle \( ABC \) we can derive Euler’s inequality \( R \geq 2r \). Indeed, in this case the inequality is equivalent to \( R^2(\alpha + \beta + \gamma) \geq \alpha \beta \gamma \). Therefore we can write

\[
R^2 \geq \frac{\alpha \beta \gamma}{\alpha + \beta + \gamma} = \frac{\alpha \beta \gamma}{2s} = \frac{4R}{2s} \cdot \frac{\alpha \beta \gamma}{4R} = \frac{2R \cdot \text{area}[ABC]}{s} = 2Rr,
\]

hence \( R \geq 2r \).

2) We can obtain the inequality

\[
R^2(\alpha + \beta + \gamma) \geq \alpha \beta \gamma
\]

by a different argument, but also by using complex numbers. This alternative proof is given in our book [1]. Indeed, with the notations in the proof of Theorem 1, we have the identity

\[
a^2(b-c) + b^2(c-a) + c^2(a-b) = (a-b)(b-c)(c-a).
\]

Passing to moduli and using the well-known triangle inequality, we obtain

\[
|a-b||b-c||c-a| \leq |a|^2|b-c| + |b|^2|c-a| + |c|^2|a-b|.
\]
Suppose that the circumcenter $O$ of triangle $ABC$ is the origin of the complex plane. Then $|a| = |b| = |c| = R$ and (5) is equivalent to inequality (3).

3) If $P$ is the centroid $G$ of triangle $ABC$, we derive the following inequality involving the medians $m_\alpha, m_\beta, m_\gamma$:

$$\frac{m_\alpha m_\beta}{\alpha \beta} + \frac{m_\beta m_\gamma}{\beta \gamma} + \frac{m_\gamma m_\alpha}{\gamma \alpha} \geq \frac{9}{4},$$

with equality if and only if triangle $ABC$ is equilateral.

Some Olympiad-calliber problems are directly connected to the result contained in Theorem 1. The first such problem deals with the case of equality when triangle $ABC$ is acute-angled.

**Problem 1.** Let $ABC$ be an acute-angled triangle and let $P$ be a point in its interior. Prove that

$$\alpha \cdot PB \cdot PC + \beta \cdot PC \cdot PA + \gamma \cdot PA \cdot PB = \alpha \beta \gamma,$$

if and only if $P$ is the orthocenter of triangle $ABC$.

(1998 Chinese Mathematical Olympiad)

**Solution.** Let $P$ be the origin of the complex plane and let $a, b, c$ be the affixes of $A, B, C$, respectively. The relation in the problem is equivalent to

$$|ab(a - b)| + |bc(b - c)| + |ca(c - a)| = |(a - b)(b - c)(c - a)|.$$

Let

$$z_1 = \frac{ab}{(a - c)(b - c)}, \quad z_2 = \frac{bc}{(b - a)(c - a)}, \quad z_3 = \frac{ca}{(c - b)(a - b)}.$$

It follows that

$$|z_1| + |z_2| + |z_3| = 1 \quad \text{and} \quad z_1 + z_2 + z_3 = 1,$$

the latter from identity (1) in the previous problem.
We will prove that $P$ is the orthocenter of triangle $ABC$ if and only if $z_1, z_2, z_3$ are positive real numbers. Indeed, if $P$ is the orthocenter, then, since the triangle $ABC$ is acute-angled, it follows that $P$ is in the interior of $ABC$. Hence there are positive real numbers $r_1, r_2, r_3$ such that

\[
\frac{a}{b - c} = -r_1i, \quad \frac{b}{c - a} = -r_2i, \quad \frac{c}{a - b} = -r_3i,
\]

implying $z_1 = r_1r_2 > 0$, $z_2 = r_2r_3 > 0$, $z_3 = r_3r_1 > 0$ and we are done. Conversely, suppose that $z_1, z_2, z_3$ are all positive real numbers. Because

\[
\frac{z_1z_2}{z_3} = \left(\frac{b}{c - a}\right)^2, \quad \frac{z_2z_3}{z_1} = \left(\frac{c}{a - b}\right)^2, \quad \frac{z_3z_1}{z_2} = \left(\frac{a}{b - c}\right)^2
\]

it follows that

\[
\frac{a}{b - c}, \quad \frac{b}{c - a}, \quad \frac{c}{a - b}
\]

are pure imaginary numbers, thus $AP \perp BC$ and $BP \perp CA$, showing that $P$ is the orthocenter of triangle $ABC$.

**Problem 2.** Let $G$ be the centroid of triangle $ABC$ and let $R_1, R_2, R_3$ be the circumradii of triangles $GBC$, $GCA$, $GAB$, respectively. Then

\[
R_1 + R_2 + R_3 \geq 3R,
\]

where $R$ is the circumradius of triangle $ABC$.

**Solution.** In Theorem 1, let $P$ be the centroid $G$ of triangle $ABC$. Then

\[
(6) \quad \alpha \cdot GB \cdot GC + \beta \cdot GC \cdot GA + \gamma \cdot GA \cdot GB \geq \alpha \beta \gamma,
\]

where $\alpha, \beta, \gamma$ are the side lengths of triangle $ABC$.

But

\[
\alpha \cdot GB \cdot GC = 4R_1 \cdot \text{area}[GBC] = 4R_1 \cdot \frac{1}{3} \text{area}[ABC]
\]

and the other two relations:

\[
\beta \cdot GC \cdot GA = 4R_2 \cdot \frac{1}{3} \text{area}[ABC], \quad \gamma \cdot GA \cdot GB = 4R_3 \cdot \frac{1}{3} \text{area}[ABC].
\]
Hence (6) is equivalent to
\[ \frac{4}{3}(R_1 + R_2 + R_3) \cdot \text{area}[ABC] \geq 4R \cdot \text{area}[ABC], \]
i.e. \( R_1 + R_2 + R_3 \geq 3R \), as desired.

**Problem 3.** Let \( ABC \) be a triangle and let \( P \) be a point in its interior. Let \( R_1, R_2, R_3 \) be the radii of the circumcircles of triangles \( PBC, PCA, PAB \), respectively. Lines \( PA, PB, PC \) intersect sides \( BC, CA, AB \) at \( A_1, B_1, C_1 \), respectively. Denote
\[ k_1 = \frac{PA_1}{AA_1}, \quad k_2 = \frac{PB_1}{BB_1}, \quad k_3 = \frac{PC_1}{CC_1}. \]

Prove that
\[ k_1 R_1 + k_2 R_2 + k_3 R_3 \geq R, \]
where \( R \) is the circumradius of triangle \( ABC \).

*(2004 Romanian IMO Team Selection Test)*

**Solution.** Note that
\[ k_1 = \frac{\text{area}[PBC]}{\text{area}[ABC]}, \quad k_2 = \frac{\text{area}[PCA]}{\text{area}[ABC]}, \quad k_3 = \frac{\text{area}[PAB]}{\text{area}[ABC]}. \]

But \( \text{area}[ABC] = \frac{\alpha \beta \gamma}{4R} \) and \( \text{area}[PBC] = \frac{\alpha \cdot PB \cdot PC}{4R_1} \). Other two similar relations for \( \text{area}[PCA] \) and \( \text{area}[PAB] \) hold.

The desired inequality is equivalent to
\[ R \frac{\alpha \cdot PB \cdot PC}{\alpha \beta \gamma} + R \frac{\beta \cdot PC \cdot PA}{\alpha \beta \gamma} + R \frac{\gamma \cdot PA \cdot PB}{\alpha \beta \gamma} \geq R, \]
which reduces to the inequality in Theorem 1.

In the case when triangle \( ABC \) is acute-angled, from Problem 1 it follows that equality holds if and only if \( P \) is the orthocenter of \( ABC \).

**Theorem 2.** Let \( P \) be an arbitrary point in the plane of triangle \( ABC \). Then
\[ \alpha \cdot PA^2 + \beta \cdot PB^2 + \gamma \cdot PC^2 \geq \alpha \beta \gamma. \]
Proof. Let us consider the origin of the complex plane at the point $P$ and let $a, b, c$ be the affixes of the vertices of triangle $ABC$. The following identity is easy to verify:

\[
\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} = 1.
\]

By passing to moduli it follows that

\[
1 = \left| \sum_{cyc} \frac{a^2}{(a-b)(a-c)} \right| \leq \sum_{cyc} \frac{|a|^2}{|a-b||a-c|}
\]

Taking into account that $|a| = PA$, $|b| = PB$, $|c| = PC$ and $|b-c| = \alpha$, $|c-a| = \beta$, $|a-b| = \gamma$, the previous inequality is equivalent to (7).

Remarks. 1) If $P$ is the circumcenter $O$ of triangle $ABC$, then $PA = PB = PC = R$ and from (8) we derive again inequality (3), which is equivalent to Euler’s inequality $R \geq 2r$.

2) If $P$ is the centroid $G$ of triangle $ABC$, then

\[
PA^2 = \frac{1}{9}[2(\beta^2 + \gamma^2) - \alpha^2], \quad PB^2 = \frac{1}{9}[2(\gamma^2 + \alpha^2) - \beta^2], \quad PC^2 = \frac{1}{9}[2(\alpha^2 + \beta^2) - \gamma^2]
\]

and (7) is equivalent to

\[
2 \sum_{cyc} (\beta^2 + \gamma^2) \geq 9 \alpha \beta \gamma + \alpha^3 + \beta^3 + \gamma^3.
\]

3) If $P$ is the incenter $I$ of triangle $ABC$, then

\[
PA = \frac{r}{\sin \frac{A}{2}}, \quad PB = \frac{r}{\sin \frac{B}{2}}, \quad PC = \frac{r}{\sin \frac{C}{2}}
\]

and is not difficult to see that we have equality in (7).

4) A different proof for (7), by using a variant of Lagrange’s identity, is given in the book [4].
Theorem 3. Let $P$ be an arbitrary point in the plane of triangle $ABC$. Then

\begin{equation}
\alpha \cdot PA^3 + \beta \cdot PB^3 + \gamma \cdot PC^3 \geq 3\alpha \beta \gamma PG,
\end{equation}

where $G$ is the centroid of triangle $ABC$.

Proof. The identity

\begin{equation}
x^3(y - z) + y^3(z - x) + z^3(x - y) = (x - y)(y - z)(z - x)(x + y + z)
\end{equation}

holds for any complex numbers $x, y, z$. Passing to moduli, we obtain

\begin{equation}
|x|^3|y - z| + |y|^3|z - x| + |z|^3|x - y| \geq |x - y||y - z||z - x||x + y + z|
\end{equation}

Let $a, b, c, z_P$ be the affixes of points $A, B, C, P$, respectively. In (12) consider $x = z_P - a$, $y = z_P - b$, $z = z_P - c$ and obtain inequality (10).

Remarks. 1) If $P$ is the circumcenter $O$ of triangle $ABC$, after some elementary transformations, (10) becomes

\begin{equation}
\frac{R^2}{6r} \geq OG.
\end{equation}

2) Squaring both sides of (13), we obtain

\begin{equation}
R^2 \geq 36r^2 \cdot OG^2.
\end{equation}

Using the relation $OG^2 = R^2 - \frac{1}{9} (\alpha^2 + \beta^2 + \gamma^2)$, (14) is equivalent to

\begin{equation}
R^2(R^2 - 4r^2) \geq 4r^2[8R^2 - (\alpha^2 + \beta^2 + \gamma^2)].
\end{equation}

The inequality (15) improves Euler’s inequality for the class of obtuse triangles. This is equivalent to proving that $\alpha^2 + \beta^2 + \gamma^2 < 8R^2$ in any such triangle. The last relation can be written as $\sin^2 A + \sin^2 B + \sin^2 C < 2$, or $\cos^2 A + \cos^2 B - \sin^2 C > 0$. That is

\begin{equation}
\frac{1 + \cos 2A}{2} + \frac{1 + \cos 2B}{2} - 1 + \cos^2 C > 0,
\end{equation}

which reduces to $\cos(A + B) \cos(A - B) + \cos^2 C > 0$. This is equivalent to $\cos C[\cos(A - B) - \cos(A + B)] > 0$, i.e. $\cos A \cos B \cos C < 0$, which is clearly true.
Bibliografie


University of Texas at Dallas  
Richardson, TX  
USA  
E-mail: titu@amc.unl.edu

"Babeș-Bolyai" University  
Faculty of Mathematics and Computer Science  
Cluj-Napoca, Romania  
E-mail: dandrica@math.ubbcluj.ro
The set of complex numbers, usually denoted as $C$ (another standard notation is $\mathbb{C}$, but I will stick to the former), is, by definition, the vector space $\mathbb{R}^2$, i.e., the set of pairs of real numbers, with operations of addition and multiplication defined as follows for any $z_1, z_2 \in C$:

\begin{align}
\overline{\overline{z}} &= z \\
\overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \quad \text{\eqref{eq:conjsum}} \\
\overline{z_1 z_2} &= \overline{z_1} \overline{z_2} \quad \text{\eqref{eq:conjprod}} \\
\overline{\frac{z_1}{z_2}} &= \frac{\overline{z_1}}{\overline{z_2}}
\end{align}

Notice that the modulus of a complex number is always a real number and in fact it will never be negative since square roots always return a positive number or zero depending on what is under the radical. Notice that if $z$ is a real number (i.e. $z = a + 0i$) then, $\sqrt{a^2} = |a|$. The triangle inequality is actually fairly simple to prove so let's do that. We'll start with the left side squared and use $\eqref{eq:zConjz}$ and $\eqref{eq:conjsum}$ to rewrite it a little.